

ORTHOGONAL BASES FOR TRANSPORTATION POLYTOPES APPLIED TO LATIN SQUARES, MAGIC SQUARES AND SUDOKU BOARDS

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ABSTRACT. We give a simple construction of an orthogonal basis for the space of $m \times n$ matrices with row and column sums equal to zero. This vector space corresponds to the affine space naturally associated with the Birkhoff polytope, contingency tables and Latin squares. We also provide orthogonal bases for the spaces underlying magic squares and Sudoku boards. Our construction combines the outer (i.e., tensor or dyadic) product on vectors with certain rooted, vector-labeled, binary trees. Our bases naturally respect the decomposition of a vector space into centrosymmetric and skew-centrosymmetric pieces; the bases can be easily modified to respect the usual matrix symmetry and skew-symmetry as well.

1. INTRODUCTION

Matrices with specified row and column sums arise in various contexts in mathematics: in the definition of the Birkhoff polytope; as statistical contingency tables; and as Latin squares, magic squares and Sudoku boards. In this paper, we give simple, explicit linear-algebraic constructions of orthogonal bases for the vector spaces underlying these families. We note from the outset that there are obvious bases for these spaces (see Section 7) that can be orthogonalized by, say, the Gram-Schmidt process. However, we doubt there is a simple closed-form description of the matrices resulting from such a process. In addition, our approach has the advantage of yielding basis vectors that respect natural decompositions of these vector spaces under various symmetries.

For $m, n \geq 2$, let $V_{m,n}$ be the $(m-1)(n-1)$ -dimensional subspace of matrices (x_{ij}) in \mathbb{R}^{mn} subject to the m requirements that the row sums are zero and n requirements that the column sums are zero:

$$(1) \quad \sum_{k=1}^n x_{ik} = 0 = \sum_{k=1}^m x_{kj}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Our main theorem in this paper is an explicit orthogonal basis for $V_{m,n}$. In Section 3 we define for each $k \geq 2$ a set $U(k)$ of $k-1$ vectors. Each element of our basis for $V_{m,n}$ can be written as an outer product of an element of $U(m)$ with an element of $U(n)$. (Recall that the *outer product* $\mathbf{u}\mathbf{w}$ of $\mathbf{u} = (u_1, \dots, u_m)$ with $\mathbf{w} = (w_1, \dots, w_n)$ is the $m \times n$ matrix whose (i, j) -th entry is $u_i w_j$.)

Theorem 1. The set of matrices $B_{m,n} = \{\mathbf{u}\mathbf{u}' : \mathbf{u} \in U(m), \mathbf{u}' \in U(n)\}$ provides an orthogonal basis for $V_{m,n}$.

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Example 2. In Section 3 it is shown that $U(3) = \{\mathbf{u}^1 = (1, -2, 1), \mathbf{u}^2 = (1, 0, -1)\}$. By Theorem 1, the following four matrices thereby form an orthogonal basis for $V_{3,3}$.

$$\mathbf{u}^1 \mathbf{u}^1 = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{u}^1 \mathbf{u}^2 = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{u}^2 \mathbf{u}^1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}, \quad \mathbf{u}^2 \mathbf{u}^2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The structure of the paper is as follows. In Section 2 we unify the objects being studied under the umbrellas of transportation polytopes and inside-out polytopes. We also explain the simple shift from the relevant affine space containing the objects of interest to the linear subspace $V_{m,n}$. In Section 3 we define the sets $U(n)$ and prove Theorem 1. In Sections 4 and 5 we modify our construction of $B_{m,n}$ so as to provide analogous bases for vector spaces of magic squares and Sudoku boards, respectively. In Section 6 we introduce simple variations of our bases that respect usual matrix symmetry and skew-symmetry. Finally, in Section 7 we explore the connection to a well-known non-orthogonal basis for $V_{m,n}$ and mention some possible directions for further study.

In order to avoid clutter, we will periodically utilize the following notations: denoting negative numbers by placing a bar of the number; replacing zeros with underscores; and omitting parentheses and commas from vectors.

2. TRANSPORTATION AND INSIDE-OUT POLYTOPES

The examples mentioned in the beginning of the Introduction are unified by the concept of a transportation (or, transport) polytope. Transportation polytopes have long been studied in the fields of statistics, mathematical programming and geometry (see the following references for different overviews: [7, 16, 6]). Let $m, n \geq 1$ and $\mathbf{r} = (r_1, \dots, r_m)$, $\mathbf{c} = (c_1, \dots, c_n)$ be two vectors of nonnegative real entries such that $\sum_{k=1}^m r_k = \sum_{k=1}^n c_k$. The vectors \mathbf{r} and \mathbf{c} are called *marginals* (or *margins* or *1-marginals*). As in Pak [13], we define the *transportation polytope* $T(\mathbf{r}, \mathbf{c})$ to be the set of $m \times n$ matrices (x_{ij}) over \mathbb{R} satisfying:

- $x_{ij} \geq 0$ for all $1 \leq i, j \leq n$,
- $\sum_{k=1}^n x_{ik} = r_i$, $1 \leq i \leq m$, and
- $\sum_{k=1}^m x_{kj} = c_j$, $1 \leq j \leq n$.

The special case of $m = n$ and $\mathbf{r} = \mathbf{c} = (1, 1, \dots, 1)$ is known as the *Birkhoff polytope* or the *polytope of doubly stochastic matrices*. The Birkhoff polytope is one of the most fundamental of polytopes — see, e.g. [17].

In many cases, one is interested solely in the lattice points \mathbb{Z}^{mn} lying in a given polytope. For example, in statistics a *contingency table* is such a lattice point in the transportation polytope $T(\mathbf{r}, \mathbf{c})$ in which the marginals consist of integers. Such tables are used to describe the distribution of a population over two variables. Knowledge of all such integer-lattice points satisfying a given set of marginals is useful in statistical tests for significance [7].

There are a number of examples where one is interested in only a subset of the lattice points lying in a polytope. The cases discussed in this paper are the following (take $m = n$).

- Let $\mathbf{r} = \mathbf{c} = (\binom{n+1}{2}, \dots, \binom{n+1}{2})$. An *order- n Latin square* is an element of $LS_n = T(\mathbf{r}, \mathbf{c})$ for which each row and column is a permutation of $\{1, 2, \dots, n\}$.
- Let $S \in \mathbb{R}_{\geq 0}$. For $\mathbf{r} = \mathbf{c} = (S, S, \dots, S)$ an *order- n semi-magic square with magic-sum S* is an element of $T(\mathbf{r}, \mathbf{c})$ for which all n^2 entries are distinct. If both main-diagonal sums also equal S , then such a matrix is not merely *semi-magic* but also *magic*. If the entries are $\{1, 2, \dots, n^2\}$ (and hence $S = n^2(n^2 + 1)/2$) then the square is said to be *normal* (terminology varies).
- An $n^2 \times n^2$ matrix naturally decomposes into n^2 submatrices, each of size $n \times n$, that simultaneously tile the entire grid. An *order- n^2 Sudoku board* is an order- n^2 Latin square

with one additional property: The entries in each of these n^2 submatrices must also be a permutation of $\{1, 2, \dots, n^2\}$.

There is an expansive literature on these families and their many variations. We direct the reader to [5, §III] for an overview of Latin squares and [4] for an overview of Magic squares and Sudoku.

In each of these three cases, the subset of lattice points of interest can be characterized in a particularly simple way. Recall that a *hyperplane arrangement* is a finite collection of hyperplanes in a vector space. A polytope in conjunction with a hyperplane arrangement is known as an *inside-out polytope*. We can describe the set of order- n Latin square using an inside-out polytopes as follows. Denote the n^2 -dimensional hypercube with sides $1 \leq x_{ij} \leq n$ by HC_n . Let A_n denote the hyperplane arrangement

$$\bigcup_{\substack{j=1 \\ j \neq k}}^n \{x_{ij} = x_{ik}\} \bigcup_{\substack{i=1 \\ i \neq j}}^n \{x_{ik} = x_{jk}\}.$$

Then the set of order- n Latin squares is the set of those lattice points lying in the polytope $\text{LS}_n \cap \text{HC}_n$ and avoiding the hyperplane arrangement A_n . The incorporation of HC_n in the definition ensures that all coordinates are between 1 and n . Avoidance of A_n thereby ensures by the pigeonhole principle that each row and column is a permutation of $\{1, 2, \dots, n\}$.

By suitably modifying the hyperplanes and polytopes considered, Sudoku boards and magic squares can also be realized as lattice points in inside-out polytopes. We omit the details and refer the reader to [2] for the general theory of inside-out polytopes.

We now discuss the relationship between the affine spaces in which our polytopes typically lie and the associated linear subspaces of \mathbb{R}^{n^2} . For concreteness, we focus on Latin squares. We begin with a well-known fact stated in the introduction. (Proofs can be found in, for example, [6, Lemma 2.3] and [14]); we include a proof here only for completeness.)

Proposition 3. For $m, n \geq 1$, the dimension of $V_{m,n}$ is $(m-1)(n-1)$.

Proof. $V_{m,n}$ is a subspace of \mathbb{R}^{mn} that is defined by $m+n$ linear equations: m of the form $\sum_k x_{ik} = 0$ and n of the form $\sum_k x_{kj} = 0$. The sum of the first m expressions equals the sum of the last n expressions, so we know there is a dependency. However, if we omit the $(m+1)$ -st equation and order the x_{ij} lexicographically, we see that the pivot columns of the remaining $m+n-1$ equations are distinct. It follows that $\dim(V_{m,n}) = mn - (m+n-1)$ as desired. \square

We have realized Latin squares as points in an n^2 -dimensional vector space. However, Latin squares actually live in a proper subspace. By definition, the polytope LS_n consists of points lying on the $2n$ hyperplanes determined by the marginals. By the argument of Proposition 3, there are exactly $2n-1$ *independent* conditions among these requirements. It follows that the polytope LS_n , and hence order- n Latin squares, live inside a $n^2 - (2n-1) = (n-1)^2$ -dimensional affine subspace of \mathbb{R}^{n^2} .

The space $V_{n,n}$ does not contain any Latin squares as the row and column sums of elements in $V_{n,n}$ are forced to be zero. However, suitable analogues of Latin squares that live in this vector space are given as follows. Note that $1 + 2 + \dots + n = \binom{n+1}{2}$. So if we subtract $\frac{1}{n}\binom{n+1}{2} = \frac{n+1}{2}$ from each element of a normal Latin square, we get a matrix in which each row and column is a permutation of $\{i - \frac{n+1}{2} : 1 \leq i \leq n\}$. These will be termed *zeroed Latin squares*.

Remark 4. A similar procedure can be used to translate the affine subspace containing an arbitrary transportation polytope to a linear subspace of $V_{m,n}$. If the marginals are initially \mathbf{r} and \mathbf{c} , then the corresponding translated polytope imposes the requirement that $x_{ij} \geq -r_i c_j / (\sum_{i=1}^m r_i)$ rather than $x_{ij} \geq 0$.

It will be useful in our discussion of magic squares in Section 4 to augment $V_{n,n}$. Let J_n be the $n \times n$ matrix of all 1's. Then the affine space containing LS_n (and hence order- n Latin squares)

is an affine subspace of the $((n-1)^2 + 1)$ -dimensional vector space $\langle J_n \rangle \oplus V_{n,n}$. Note that in this space, Latin squares will have the extra coordinate equal to $\frac{n+1}{2}$.

3. ORTHOGONAL BASES

For each $n \geq 2$ we will define a set of $n-1$ mutually orthogonal vectors $U(n) = \{\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^{n-1}\}$. Our first ingredient will be a vector-valued function, \mathbf{w} , on the integers greater than 1. The second ingredient will be a (rooted), labeled binary tree, T_n , for each n . The desired vectors \mathbf{u}^i will be the labels of the vertices of our tree T_n .

For each $n \geq 2$ we now define $\mathbf{w}(n) = (w(n)_0, w(n)_1, \dots, w(n)_{n-1})$ as follows. If n is odd, then set $w(n)_i = (n-1)/2$ for i odd and $-(n+1)/2$ for i even. Set $\mathbf{w}(2) = (1, -1)$ and $\mathbf{w}(4) = (1, -1, -1, 1)$. For n even and greater than 4, say $n = 2m$, set $w(n)_i = w(m)_{i \pmod m}$ for $0 \leq i \leq n-1$. Table 1 shows the values of \mathbf{w} for small n .

TABLE 1. Values of \mathbf{w} function (negatives are denoted by bars).

| n | $\mathbf{w}(n)$ | n | $\mathbf{w}(n)$ |
|-----|---|-----|---|
| 3 | 1 $\bar{2}$ 1 | 4 | 1 $\bar{1}$ $\bar{1}$ 1 |
| 5 | 2 $\bar{3}$ 2 $\bar{3}$ 2 | 6 | 1 $\bar{2}$ 1 1 $\bar{2}$ 1 |
| 7 | 3 $\bar{4}$ 3 $\bar{4}$ 3 $\bar{4}$ 3 | 8 | 1 $\bar{1}$ $\bar{1}$ 1 1 $\bar{1}$ $\bar{1}$ 1 |
| 9 | 4 $\bar{5}$ 4 $\bar{5}$ 4 $\bar{5}$ 4 $\bar{5}$ 4 | 10 | 2 $\bar{3}$ 2 $\bar{3}$ 2 2 $\bar{3}$ 2 $\bar{3}$ 2 |
| 11 | 5 $\bar{6}$ 5 $\bar{6}$ 5 $\bar{6}$ 5 $\bar{6}$ 5 $\bar{6}$ 5 | 12 | 1 $\bar{2}$ 1 1 $\bar{2}$ 1 1 $\bar{2}$ 1 1 $\bar{2}$ 1 |

Given $n \geq 2$, we construct T_n iteratively by depth (i.e., distance from the root), starting with a root labeled $\mathbf{w}(n)$. Once all vertices of depth at most d have been identified and labeled, we construct vertices at depth $d+1$ as follows: Suppose a vertex at depth d has label $\mathbf{u} = (u_0, u_1, \dots, u_{n-1})$. Let $I_+(\mathbf{u}) = \{i_0 < i_1 < \dots < i_{a'-1}\}$ be the indices for which $u_i > 0$ and $I_-(\mathbf{u}) = \{j_0 < j_1 < \dots < j_{a''-1}\}$ be those indices for which $u_j < 0$. If $|I_+(\mathbf{u})| = a'$ is at least 2, then we attach a left child with label $\mathbf{u}' = (u'_0, u'_1, \dots, u'_{n-1})$ where

$$(2) \quad u'_i = \begin{cases} 0, & \text{for } i \notin I_+(\mathbf{u}), \\ w(a')_r, & \text{for } i = i_r. \end{cases}$$

Similarly, if $|I_-(\mathbf{u})| = a''$ is at least 2, we attach a right child $\mathbf{u}'' = (u''_0, u''_1, \dots, u''_{n-1})$ where

$$(3) \quad u''_i = \begin{cases} 0, & \text{for } i \notin I_-(\mathbf{u}), \\ w(a'')_r, & \text{for } i = j_r. \end{cases}$$

If $I_+(\mathbf{u}) = I_-(\mathbf{u}) = 1$, then \mathbf{u} is a leaf.

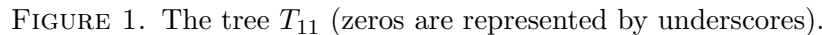
Note that the vector formed by the nonzero entries of any vertex \mathbf{u} of T_n equals $\mathbf{w}(k)$ for some k . If \mathbf{u} corresponds to $\mathbf{w}(2)$ after ignoring zeros, then we say that \mathbf{u}^i is *skew-symmetric* since $u_j^i = -u_j^i$ for all j . In all other cases, $u_j^i = u_{n-1-j}^i$ for all j and we refer to such \mathbf{u}^i as *symmetric*.

Lemma 5. For $n \geq 2$, T_n has $n-1$ vertices, $\lfloor n/2 \rfloor$ of which are skew-symmetric and $\lfloor (n-1)/2 \rfloor$ of which are symmetric.

Proof. First note that

$$\lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor = n-1,$$

regardless of the parity of n . So the claim regarding the total number of vertices of T_n follows directly from the classification of their types.


$$\left| \frac{2^k \frac{m+1}{2}}{2} \right| + \left| \frac{2^k \frac{m-1}{2}}{2} \right| = 2^{k-2}(m+1) + 2^{k-2}(m-1) = 2^{k-1}m$$
$$\left| \frac{2^k \frac{m+1}{2} - 1}{2} \right| + \left| \frac{2^k \frac{m-1}{2} - 1}{2} \right| = 2^{k-2}(m+1) - 1 + 2^{k-2}(m-1) - 1 = 2^{k-1}m - 2$$

Definition 1. Let $n \geq 2$. Define the set $U(n) = \{\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^{n-1}\}$ by setting \mathbf{u}^i to be the i -th vertex encountered while performing a depth-first traversal of T_n (choose the left child before the right child).

$$\begin{array}{ll} \mathbf{u}^1 = 5\bar{6}5\bar{6}5\bar{6}5\bar{6}5\bar{6}5, & \mathbf{u}^2 = 1_2_1_1_2_1, \\ \mathbf{u}^3 = 1_ _ _ \bar{1} _ \bar{1} _ _ 1, & \mathbf{u}^4 = 1_ _ _ _ _ _ _ \bar{1}, \\ \mathbf{u}^5 = _ _ _ 1 _ \bar{1} _ _ _ , & \mathbf{u}^6 = _ 1 _ _ _ \bar{1} _ _ \\ \mathbf{u}^7 = _ 2 _ \bar{3} _ 2 _ \bar{3} _ 2 _ , & \mathbf{u}^8 = _ 1 _ _ \bar{2} _ _ 1 _ , \\ \mathbf{u}^9 = _ 1 _ _ _ _ \bar{1} _ _ , & \mathbf{u}^{10} = _ _ 1 _ _ \bar{1} _ _ . \end{array}$$
$$\begin{aligned} (1) \quad & \sum_{i=0}^{n-1} u_i = 0 \text{ and} \\ (2) \quad & |\{u_i > 0\}| = |\{u_i < 0\}| = 1. \end{aligned}$$

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For the second part, note that each \mathbf{u} equals $\mathbf{w}(k)$ for some k once zeros are ignored. That each $\mathbf{w}(k)$ has a unique positive value and a unique negative value follows by induction. \square

Proposition 7. The set $U(n)$ is an orthogonal set.

Proof. Consider vectors $\mathbf{u}^j, \mathbf{u}^k \in U(n)$. Without loss of generality we assume $j < k$. If \mathbf{u}^j does not lie on the path from \mathbf{u}^k to the root, then the orthogonality follows trivially since for each index $0 \leq i \leq n-1$, either $u_i^j = 0$ or $u_i^k = 0$ (or both). So suppose \mathbf{u}^j does lie on the path from \mathbf{u}^k to the root. It follows from the definitions of I_+ , I_- , Lemma 6.2, and equations (2) and (3) that for all pairs $u_i^k, u_{i'}^k \neq 0$, we have $u_i^j = u_{i'}^j$. But then $\mathbf{u}^j \cdot \mathbf{u}^k$ is a scalar multiple of $\sum_{i=0}^{n-1} u_i^k$, which by Lemma 6.1 is zero. \square

The *outer product* of two vectors $\mathbf{a} = (a_0, a_1, \dots, a_{m-1})$, $\mathbf{b} = (b_0, b_1, \dots, b_{n-1})$ is given by

$$\mathbf{ab} = \mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_0 b_0 & a_0 b_1 & \dots & a_0 b_{n-1} \\ a_1 b_0 & a_1 b_1 & \dots & a_1 b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1} b_0 & a_{m-1} b_1 & \dots & a_{m-1} b_{n-1} \end{bmatrix}.$$

This product, which we denote by juxtaposition, can be viewed (treating \mathbf{a} and \mathbf{b} as row vectors) as a matrix product $\mathbf{a}^T \mathbf{b}$ or as a *dyadic product*. We are now ready to prove Theorem 1.

Proof of Theorem 1. Note that each row and column of $\mathbf{u}^i \mathbf{u}^j$ is a scalar multiple of a vector whose entries sum to zero. It follows immediately that each element of $B_{m,n}$ lies in $V_{m,n}$.

We first show that the cardinality of the set $B_{m,n}$ is $(m-1)(n-1)$, the dimension of $V_{m,n}$. So suppose that $\mathbf{u}^i \mathbf{u}^j = \mathbf{u}^k \mathbf{u}^\ell$ for some $1 \leq i, k \leq m-1$ and $1 \leq j, \ell \leq n-1$. Then, in particular, the a -th row of $\mathbf{u}^i \mathbf{u}^j$ equals the a -th row of $\mathbf{u}^k \mathbf{u}^\ell$ for each $1 \leq a \leq m$. This implies in turn that

$$(4) \quad \mathbf{u}_a^i \mathbf{u}^j = (u_a^i u_0^j, u_a^i u_1^j, \dots, u_a^i u_{n-1}^j) \text{ equals } \mathbf{u}_a^k \mathbf{u}^\ell = (u_a^k u_0^\ell, u_a^k u_1^\ell, \dots, u_a^k u_{n-1}^\ell).$$

Since the $U(n)$ is an orthogonal set by Proposition 7, it follows that either $u_a^i = u_a^k = 0$ or that $\mathbf{u}^j = \mathbf{u}^\ell$. Since \mathbf{u}^i is not the zero vector, it follows that we can pick an a for which $u_a^i \neq 0$. We conclude that $\mathbf{u}^j = \mathbf{u}^\ell$ and hence that $j = \ell$ by orthogonality. In turn, this tells us that $u_a^i = u_a^k$ for all a . Again by Proposition 7, we conclude $i = k$. We conclude that the $(m-1)(n-1)$ products $\mathbf{u}^i \mathbf{u}^j$ are all distinct.

It follows immediately from the definition of the $\mathbf{u}^i \mathbf{u}^j$ and Proposition 7 that none of the $\mathbf{u}^i \mathbf{u}^j$ are the zero vector. Hence, to show linear independence, it suffices to show that they are pairwise orthogonal. Since we know the dimension of $V_{m,n}$ to be $(m-1)(n-1)$, we can then conclude that the elements of $B_{m,n}$ form a basis, as desired. So: Let $1 \leq i, k \leq m-1$ and $1 \leq j, \ell \leq n-1$. Then

$$\mathbf{u}^i \mathbf{u}^j \cdot \mathbf{u}^k \mathbf{u}^\ell = \sum_{a=1}^m \sum_{b=1}^n (\mathbf{u}^i \mathbf{u}^j)_{a,b} (\mathbf{u}^k \mathbf{u}^\ell)_{a,b} = \sum_{a=1}^m u_a^i u_a^k \sum_{b=1}^n u_b^j u_b^\ell = (\mathbf{u}^i \cdot \mathbf{u}^k) (\mathbf{u}^j \cdot \mathbf{u}^\ell).$$

By the orthogonality of the \mathbf{u}^i , we conclude immediately that $\mathbf{u}^i \mathbf{u}^j$ and $\mathbf{u}^k \mathbf{u}^\ell$ are orthogonal whenever $(i, j) \neq (k, \ell)$. This completes the proof. \square

| | | |
|----|----|----|
| 1 | -1 | 0 |
| -1 | 0 | 1 |
| 0 | 1 | -1 |

Example 8. The zeroed Latin square can be written as a linear combination of two of

the elements of $B_{3,3}$:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{2} (\mathbf{u}^1 \mathbf{u}^2 + \mathbf{u}^2 \mathbf{u}^1) = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}.$$

Table 2 lists the expansions for all twelve order-3 zeroed Latin squares.

TABLE 2. Expansions of order-3 zeroed Latin squares in the basis $B_{3,3}$.

| u^1u^1 | | | | u^1u^2 | u^2u^1 | u^2u^2 | u^1u^1 | | | | u^1u^2 | u^2u^1 | u^2u^2 | |
|----------|----|----|--|----------|----------|----------|----------------|----------------|---------------|----------------|----------------|----------------|----------------|----------------|
| 1 | -1 | 0 | | | | | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| -1 | 0 | 1 | | | | | | | | | | | | |
| 0 | 1 | -1 | | | | | | | | | | | | |
| 0 | -1 | 1 | | | | | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| 1 | 0 | -1 | | | | | | | | | | | | |
| -1 | 1 | 0 | | | | | | | | | | | | |
| 0 | -1 | 1 | | | | | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{3}{4}$ |
| -1 | 1 | 0 | | | | | | | | | | | | |
| 1 | 0 | -1 | | | | | | | | | | | | |
| 1 | -1 | 0 | | | | | $\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{3}{4}$ |
| 0 | 1 | -1 | | | | | | | | | | | | |
| -1 | 0 | 1 | | | | | | | | | | | | |
| 1 | 0 | -1 | | | | | $-\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{3}{4}$ |
| 0 | -1 | 1 | | | | | | | | | | | | |
| -1 | 1 | 0 | | | | | | | | | | | | |
| -1 | 0 | 1 | | | | | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{3}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{3}{4}$ |
| 1 | -1 | 0 | | | | | | | | | | | | |
| 0 | 1 | -1 | | | | | | | | | | | | |

The transportation polytopes defined at the beginning of this section are sometimes referred to as *2-way transportation polytopes*. There are multiple ways to generalize to dimensions $d > 2$ by placing various constraints on $p_1 \times p_2 \times \cdots \times p_d$ arrays of real numbers. In the case where all of the 1-marginals (i.e., the sums over all but one index) are specified, we obtain an affine, d -dimensional analogue V_{p_1, p_2, \dots, p_d} of $V_{m, n}$ whose dimension is $\prod_{i=1}^d (p_i - 1)$. Arguments analogous to those given in this section show that the d -fold products $u^{i_1} u^{i_2} \cdots u^{i_d}$ give an orthogonal basis for V_{p_1, p_2, \dots, p_d} .

4. MAGIC SQUARES

In analogy with our terminology for Latin squares, we will use the term *zeroed magic square* to refer to a magic square for which all row, column and main-diagonal sums are 0.

Zeroed magic squares lie in a codimension-2 subspace of $V_{n, n}$ obtained by imposing the two additional constraints that $\sum_{i=1}^n x_{i, i} = 0 = \sum_{i=1}^n x_{i, n-i+1}$. (We leave it to the reader to check that these conditions are independent of each other and of the Latin square conditions.) Let \bar{V}_n denote this codimension-2 subspace.

Lemma 9. If $u^i, u^j \in U(n)$, $i \neq j$, then $u^i u^j \in \bar{V}_n$.

Proof. This follows by an argument analogous to that found in the proof of Proposition 7. If \mathbf{u}^i and \mathbf{u}^j are not related in T_n , then all diagonal and anti-diagonal entries of $\mathbf{u}^i \mathbf{u}^j$ are zero. Otherwise, both the diagonal $(x_{1,1}, x_{2,2}, \dots, x_{n,n})$ and anti-diagonal $(x_{n,1}, x_{n-1,2}, \dots, x_{1,n})$ are scalar multiples of either \mathbf{u}^i or \mathbf{u}^j (depending on which is closer to the root). Since the sum of the entries of each \mathbf{u}^k is 0 by Lemma 6.1, the result follows. \square

In light of the above lemma, we will construct a basis for \overline{V}_n by taking $\{\mathbf{u}^i \mathbf{u}^j : i < j\}$ and adjoining $(n-1)-2$ vectors generated from the $n-1$ vectors of the form $\mathbf{u}^i \mathbf{u}^i$. Write $k = \lfloor n/2 \rfloor$, $k' = \lfloor (n-1)/2 \rfloor$, and (recalling the definition from Section 3 and Lemma 5), write $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^k\}$ for the set $\{\mathbf{u}^i \mathbf{u}^i : \mathbf{u}^i \text{ is skew-symmetric}\}$; write $\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^{k'}\}$ for the set $\{\mathbf{u}^i \mathbf{u}^i : \mathbf{u}^i \text{ is symmetric}\}$.

For $1 \leq i \leq k-1$, define

$$\overline{\mathbf{x}}^i = \sum_{j=1}^k u_j^{k;i} \mathbf{x}^j.$$

Since the nonzero diagonal entries of the \mathbf{x}^j are all 1's and the nonzero anti-diagonal entries of the \mathbf{x}^j are all -1 's, it follows immediately from Lemma 6.1 that each $\overline{\mathbf{x}}^i$ lies in \overline{V}_n . Linear independence of these $k-1$ vectors will follow from the orthogonality arguments contained in the proof of Theorem 10.

We can proceed similarly in finding a codimension-1 subspace of the span of the \mathbf{y}^i , except we need to account for the fact that the diagonal sums of the \mathbf{y}^i vary. Let

$$\ell_i = \sum_{j=1}^n \mathbf{y}_{jj}^i, 1 \leq i \leq k' \text{ and } \ell = \text{lcm}\{\ell_1, \dots, \ell_{k'}\}.$$

Then for $1 \leq i \leq k'-1$ we set

$$\overline{\mathbf{y}}^i = \sum_{j=1}^{k'} \frac{\ell}{\ell_j} u_j^{k';i} \mathbf{y}^j.$$

Theorem 10. The set

$$(5) \quad \{\mathbf{u}^i \mathbf{u}^j : i < j\} \cup \{\overline{\mathbf{x}}^i\}_{i=1}^{k-1} \cup \{\overline{\mathbf{y}}^i\}_{i=1}^{k'-1}$$

is an orthogonal basis for \overline{V}_n .

Proof. We already know that $B_{n,n}$ is an orthogonal set. Note that each $\overline{\mathbf{x}}^i$ is a linear combination of elements from $\{\mathbf{x}^1, \dots, \mathbf{x}^k\}$ and each $\overline{\mathbf{y}}^j$ is a linear combination of elements from the disjoint set $\{\mathbf{y}^1, \dots, \mathbf{y}^{k'}\}$. To prove orthogonality of the entire set, it therefore suffices to show that the $\overline{\mathbf{x}}^i$ are mutually orthogonal and that the $\overline{\mathbf{y}}^j$ are mutually orthogonal. Since we will have identified in equation (5) $(n-1)^2 - 2$ linearly independent vectors in a $((n-1)^2 - 2)$ -dimensional vector space, the claim will follow.

We have

$$\begin{aligned} \overline{\mathbf{x}}^i \cdot \overline{\mathbf{x}}^j &= \left(\sum_{a=1}^k u_a^{k;i} \mathbf{x}^a \right) \cdot \left(\sum_{b=1}^k u_b^{k;j} \mathbf{x}^b \right) \\ &= \sum_{a=1}^k \|\mathbf{x}^a\|^2 u_a^{k;i} u_a^{k;j} \\ &= 4(\mathbf{u}^{k;i} \cdot \mathbf{u}^{k;j}) = 4\delta_{i,j}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta. So $\{\overline{\mathbf{x}}^1, \dots, \overline{\mathbf{x}}^{k-1}\}$ is an orthogonal set and, since each $\overline{\mathbf{x}}^i$ is easily seen to be nonzero, it follows that it is a linearly independent set. To prove the analogous result

for the $\bar{\mathbf{y}}^i$ we first note that for each \mathbf{y}^i , there exists a $\mathbf{u} \in U(n)$ such that $\mathbf{y}^i = \mathbf{u}\mathbf{u}$. We also note that

$$\ell_i = \sum_{j=1}^n \mathbf{y}_{jj}^i = \sum_{j=1}^n (\mathbf{u}_j)^2 = \|\mathbf{u}\|^2 = \sqrt{\|\mathbf{u}\|^2 \|\mathbf{u}\|^2} = \sqrt{\sum_{a=1}^n u_a^2 \sum_{b=1}^n u_b^2} = \sqrt{\sum_{a,b=1}^n (u_a u_b)^2} = \|\mathbf{y}^i\|.$$

Hence,

$$\begin{aligned} \bar{\mathbf{y}}^i \cdot \bar{\mathbf{y}}^j &= \left(\sum_{a=1}^{k'} \frac{\ell}{\ell_a} u_a^{k';i} \mathbf{y}^a \right) \cdot \left(\sum_{b=1}^{k'} \frac{\ell}{\ell_b} u_b^{k';j} \mathbf{y}^b \right) \\ &= \sum_{a=1}^{k'} \frac{\ell^2}{\ell_a^2} u_a^{k';i} u_a^{k';j} \|\mathbf{y}^a\|^2 = \ell^2 \sum_{a=1}^{k'} u_a^{k';i} u_a^{k';j} = \ell^2 (\mathbf{u}^{k';i} \cdot \mathbf{u}^{k';j}) = \ell^2 \delta_{i,j}. \end{aligned}$$

□

Example 11. For $n = 3$,

$$\bar{V}_3 = \langle \mathbf{u}^1 \mathbf{u}^2, \mathbf{u}^2 \mathbf{u}^1 \rangle = \left\langle \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix} \right\rangle.$$

Example 12. Consider $n = 6$. We have

$$U(6) = \{(1, -2, 1, 1, -2, 1), (1, 0, -1, -1, 0, 1), (1, 0, 0, 0, 0, -1), (0, 0, 1, -1, 0, 0), (0, 1, 0, 0, -1, 0)\}.$$

So \mathbf{u}^1 and \mathbf{u}^2 are symmetric while $\mathbf{u}^3, \mathbf{u}^4$ and \mathbf{u}^5 are skew-symmetric. It follows that

$$\mathbf{x}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{x}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{y}^1 = \begin{bmatrix} 1 & -2 & 1 & 1 & -2 & 1 \\ -2 & 4 & -2 & -2 & 4 & -2 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ -2 & 4 & -2 & -2 & 4 & -2 \\ 1 & -2 & 1 & 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{y}^2 = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

We find from considering $\mathbf{u}^{3;1} = (1, -2, 1)$ and $\mathbf{u}^{3;2} = (1, 0, -1)$ that

$$\bar{\mathbf{x}}^1 = \mathbf{x}^1 - 2\mathbf{x}^2 + \mathbf{x}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{x}}^2 = \mathbf{x}^1 - \mathbf{x}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, after computing $\ell_1 = 12, \ell_2 = 4, \ell = \text{lcm}(12, 4) = 12$ and $\mathbf{u}^{2;1} = (1, -1)$, we find that

$$\bar{\mathbf{y}}^1 = \frac{12}{12} \cdot 1 \cdot \mathbf{y}^1 + \frac{12}{4} \cdot (-1) \cdot \mathbf{y}^2 = \begin{bmatrix} -2 & -2 & 4 & 4 & -2 & -2 \\ -2 & 4 & -2 & -2 & 4 & -2 \\ 4 & -2 & -2 & -2 & -2 & 4 \\ 4 & -2 & -2 & -2 & -2 & 4 \\ -2 & 4 & -2 & -2 & 4 & -2 \\ -2 & -2 & 4 & 4 & -2 & -2 \end{bmatrix}.$$

5. SUDOKU

Let Sud_{n^2} denote the subspace of V_{n^2, n^2} arising from requiring that the n^2 $n \times n$ submatrices that tile an $n^2 \times n^2$ matrix all sum to zero. Zeroed Sudoku boards are certain points of \mathbb{Z}^{n^4} lying in Sud_{n^2} .

Proposition 13. The dimension of Sud_{n^2} is $n^4 - (2n^2 - 1) - (n - 1)^2 = n(n - 1)^2(n + 2) = n^2(n - 1)^2 + 2n(n - 1)^2$.

Proof. Order the n^4 variables in an $n^2 \times n^2$ matrix by reading rows from left to right, starting with the top row and working towards the bottom. As in the proof of Proposition 3, omit the condition corresponding to the first column. This leaves $2n^2 - 1$ independent row/column conditions. Consider an $n \times n$ square flush with the top edge. That its entries sum to zero follows from the conditions on the $(n - 1)$ squares lying below it along with the conditions on its $n - 1$ columns. That the condition on any $n \times n$ square flush with the leftmost column is redundant follows similarly. By removing these $2n - 1$ conditions, we are left with $(n - 1)^2$ conditions on the $n \times n$ squares. The remaining conditions have mutually distinct pivot columns, so must be linearly independent. \square

Let \mathbf{e}^i be the (length- n) vector of all zeros except for a 1 in position i . Let \mathbf{f} be the length n vector of all 1's.

Theorem 14. A basis for Sud_{n^2} is given by

$$(6) \quad \{\mathbf{e}^i \mathbf{e}^j \otimes \mathbf{u}^k \mathbf{u}^\ell : 1 \leq i, j \leq n, 1 \leq k, \ell \leq n - 1\} \cup \\ \{\mathbf{u}^i \mathbf{e}^j \otimes \mathbf{f} \mathbf{u}^k : 1 \leq i, k \leq n - 1, 1 \leq j \leq n\} \cup \\ \{\mathbf{e}^j \mathbf{u}^i \otimes \mathbf{u}^k \mathbf{f} : 1 \leq i, k \leq n - 1, 1 \leq j \leq n\}.$$

Proof. For ease of reference, refer to the three sets in equation (6) as A , B and C , respectively. Each vector listed is manifestly non-zero. To show linear independence, it therefore suffices to show that the vectors are pairwise orthogonal. And since the first set yields $n^2(n - 1)^2$ vectors while the second and third each yield $n(n - 1)^2$, once linear independence is shown, that the set is spanning will follow automatically from our dimension count in Proposition 13.

Consider two arbitrary, distinct vectors from equation (6). Our proof of orthogonality is broken into six parts according to the which of the sets A , B or C these vectors live in. For the reader's convenience we illustrate in equation (7) one example matrix from each of the sets A , B and C .

- (1) *Both in A.* Consider the dot product of $\mathbf{e}^i \mathbf{e}^j \otimes \mathbf{u}^k \mathbf{u}^\ell$ and $\mathbf{e}^{i'} \mathbf{e}^{j'} \otimes \mathbf{u}^{k'} \mathbf{u}^{\ell'}$. If $i \neq i'$ or $j \neq j'$, then each of the n^4 coordinates is 0 for at least one of the vectors. If $i = i'$ and $j = j'$, then we are reduced to checking orthogonality in $B_{n,n}$, which we have already done in Theorem 1.
- (2) *Both in B.* Consider the dot product of $\mathbf{u}^i \mathbf{e}^j \otimes \mathbf{f} \mathbf{u}^k$ and $\mathbf{u}^{i'} \mathbf{e}^{j'} \otimes \mathbf{f} \mathbf{u}^{k'}$. If $j \neq j'$, then there are no nonzero entries in common, so assume $j = j'$. The presence of \mathbf{f} , as far as the dot product is concerned, simply multiplies the final result by n . So we are reduced to considering the dot product of $\mathbf{u}^i \mathbf{u}^k$ and $\mathbf{u}^{i'} \mathbf{u}^{k'}$. This is known to be $\delta_{(i,j),(i',k')}$ by Theorem 1.
- (3) *Both in C.* By taking the transpose of each matrix, this reduces to the previous case.
- (4) *One in A, one in B.* Consider the dot product of $\mathbf{e}^i \mathbf{e}^j \otimes \mathbf{u}^k \mathbf{u}^\ell$ and $\mathbf{u}^{i'} \mathbf{e}^{j'} \otimes \mathbf{f} \mathbf{u}^{k'}$. If $j' \neq j$ or $u_{i'}^{j'} = 0$, then the result is immediately zero. Otherwise, we are reduced to considering the dot product of $\mathbf{u}^k \mathbf{u}^\ell$ and $\mathbf{f} \mathbf{u}^{k'}$. Since the set $U(n)$ is orthogonal, the dot product will be zero unless $\ell = k'$. In this case, the result will be $n \|\mathbf{u}^\ell\|^2 \sum_a u_a^k = 0$.
- (5) *One in A, one in C.* By taking the transpose of each matrix, this reduces to the previous case.

- (6) *One in B, one in C.* Consider the dot product of $\mathbf{u}^i \mathbf{e}^j \otimes \mathbf{f} \mathbf{u}^k$ and $\mathbf{e}^{i'} \mathbf{u}^{j'} \otimes \mathbf{u}^{k'} \mathbf{f}$. The result is immediately zero if $u_{i'}^i = 0$ or $u_j^{j'} = 0$. Otherwise, the problem reduces to the dot product of $\mathbf{f} \mathbf{u}^k$ and $\mathbf{u}^{k'} \mathbf{f}$. This is easily computed as

$$\sum_{a,b} (\mathbf{f} \mathbf{u}^i)_{a,b} (\mathbf{u}^j \mathbf{f})_{a,b} = \sum_{a,b} u_b^i u_a^j = \sum_b u_b^i \sum_a u_a^j = 0$$

by Lemma 1.

This completes the proof. \square

Example 15. Let $n = 3$. Below we illustrate a basis element arising from each of the three sets of equation (6).

$$(7) \quad \mathbf{e}^1 \mathbf{e}^2 \otimes \mathbf{u}^2 \mathbf{u}^2 = \left[\begin{array}{c|cc|c} 0 & 1 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 \end{array} \right], \mathbf{u}^2 \mathbf{e}^2 \otimes \mathbf{f} \mathbf{u}^1 = \left[\begin{array}{c|cc|c} 0 & 1 & -2 & 1 & 0 \\ \hline 0 & 1 & -2 & 1 & 0 \\ \hline 0 & 1 & -2 & 1 & 0 \end{array} \right], \text{ and}$$

$$\mathbf{e}^1 \mathbf{u}^1 \otimes \mathbf{u}^1 \mathbf{f} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\ -2 & -2 & -2 & 4 & 4 & 4 & -2 & -2 & -2 \\ 1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

6. SYMMETRIES OF THE BASES

An $n \times n$ matrix $A = (a_{ij})$ is *centrosymmetric* if $a_{ij} = a_{n-i+1, n-j+1}$ for all i, j . It is *skew-centrosymmetric* if $a_{ij} = -a_{n-i+1, n-j+1}$ for all i, j .

Lemma 16. Let

$$\begin{aligned} \text{CS}_n &= \{\mathbf{v} \in \mathbf{V}_{n,n} : \mathbf{v} \text{ is centrosymmetric}\} \text{ and} \\ \text{SCS}_n &= \{\mathbf{v} \in \mathbf{V}_{n,n} : \mathbf{v} \text{ is skew-centrosymmetric}\}. \end{aligned}$$

Then $\mathbf{V}_{n,n} = \text{CS}_n \oplus \text{SCS}_n$.

Proof. The proof relies on the same technique used to show that any space of matrices splits into symmetric and skew-symmetric parts. Define a “rotation-by-180-degrees” map $\theta : \mathbf{V}_{n,n} \rightarrow \mathbf{V}_{n,n}$ by sending the matrix $A = (a_{ij}) \in \mathbf{V}_{n,n}$ to $\theta(A) = (a_{n-i+1, n-j+1})$. Then the matrix $\text{cs}(A) = (A + \theta(A))/2 \in \text{CS}_n$ and $\text{scs}(A) = (A - \theta(A))/2 \in \text{SCS}_n$. Furthermore, $A = \text{cs}(A) + \text{scs}(A)$ and $\text{CS}_n \cap \text{SCS}_n$ is the singleton set consisting of the $n \times n$ zero matrix. \square

The basis $B_{n,n}$ for $\mathbf{V}_{n,n}$ naturally splits into centrosymmetric and skew-centrosymmetric pieces. More precisely, the basis vector $\mathbf{u}^i \mathbf{u}^j \in \text{CS}_n$ if and only if either both \mathbf{u}^i and \mathbf{u}^j are symmetric or if neither are. We can decompose $\mathbf{V}_{n,n}$ further by considering the symmetric and skew-symmetric

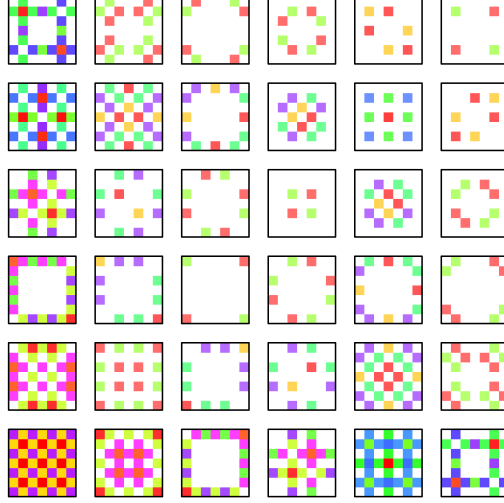


FIGURE 2. Basis for $V_{7,7}$ that has been decomposed in symmetric and skew-symmetric parts. Numbers have been replaced with colors in order to highlight the various symmetries.

parts of matrices. For example, we can replace each pair of basis vectors $\{\mathbf{u}^i \mathbf{u}^j, \mathbf{u}^j \mathbf{u}^i\}$ for $i \neq j$ with the pair

$$\left\{ \frac{\mathbf{u}^i \mathbf{u}^j + (\mathbf{u}^i \mathbf{u}^j)^T}{2}, \frac{\mathbf{u}^i \mathbf{u}^j - (\mathbf{u}^i \mathbf{u}^j)^T}{2} \right\}.$$

Example 17. If the above replacements are performed on $B_{3,3}$, we obtain the basis

$$\left\{ \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right\}.$$

Note that the first and last are symmetric matrices lying in CS_n ; the second and third both lie in SCS_n , but the second is symmetric while the third is skew-symmetric. The subspace of $V_{3,3}$ consisting of matrices that are both centrosymmetric and skew-symmetric is zero-dimensional. Also, note that the second and third basis vectors are, in fact, zeroed Latin squares. Figure 2 illustrates the analogous basis for $n = 7$.

7. FUTURE DIRECTIONS

As mentioned in the introduction, there is a simple, non-orthogonal basis for $V_{m,n}$. For $1 \leq a \leq m-1$ and $1 \leq b \leq n-1$, let $F_{a,b} = (f_{ij})$ be the $m \times n$ matrix of all zeros except for $f_{a,b} = f_{a+1,b+1} = 1$ and $f_{a+1,b} = f_{a,b+1} = -1$. It is trivial to see that each $F_{a,b}$ lies in $V_{m,n}$. Also, as each of the $(m-1)(n-1)$ matrices $F_{a,b}$ has a distinct “northwest corner”, it follows that they are linearly independent and hence a basis. They are not, in general, orthogonal. However, they do satisfy the important properties required by a *Markov basis*. Roughly: Fix marginals \mathbf{r} and \mathbf{c} and define a graph $G(\mathbf{r}, \mathbf{c})$ whose vertices are all contingency tables in $T(\mathbf{r}, \mathbf{c})$. Add an edge between two vertices differing by $\pm F_{a,b}$. The resulting graph can be shown to be connected. It turns out that one can construct a random (2-way) contingency table with given marginals by taking a random walk on $G(\mathbf{r}, \mathbf{c})$.

If we try to construct an analogous Markov chain using the elements of $B_{m,n}$ as our basis, we immediately run into a problem: Not every contingency table is a \mathbb{Z} -linear combination of the

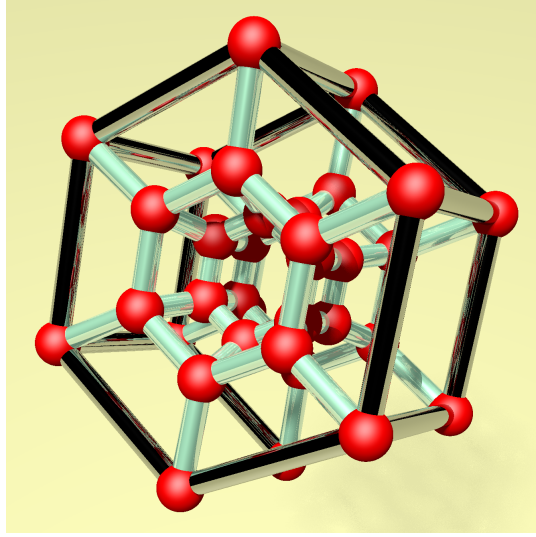


FIGURE 3. Visualization of dual of the polytope formed by the convex hull of the twelve order-3 Latin squares.

elements of $B_{m,n}$. Since the number of contingency tables with fixed marginals is finite, we can address this by suitably rescaling the elements of $B_{m,n}$. However, if we do this, then even if a given \mathbb{Z} -linear combination lies in $T(\mathbf{r}, \mathbf{c})$, it might not correspond to an actual contingency table (i.e., after performing the shift to the affine plane, it might not have integer entries).

Nonetheless, this process might be worth further exploration. A bound on the necessary amount of scaling can be found by expressing $B_{m,n}$ in terms of the basis $\{F_{a,b}\}$. Using the correspondingly scaled elements of $B_{m,n}$, a random walk could be taken and then integer programming used to find the closest contingency table. However, it is unclear whether the orthogonality of the $B_{m,n}$ is worth these complications.

Question 18. Is there any benefit to constructing a Markov chain based on the elements of the orthogonal basis $B_{m,n}$ rather than on the (non-orthogonal) basis $\{F_{a,b}\}$.

7.1. The polytope of Latin squares. Viewing any zeroed Latin square as a vector in \mathbb{R}^{n^2} , its norm squared is the square pyramidal number $n \sum_{i=1}^n i^2 = n^2(n^2 - 1)/12$. Let $\bar{B}_{n,n}$ be the orthonormal basis for $V_{n,n}$ obtained by normalizing the elements of $B_{n,n}$. It follows that the coordinates (c_{ij}) of any zeroed Latin square with respect to $\bar{B}_{n,n}$ lie on a sphere in $V_{n,n}$ centered at the origin whose radius squared is $n^2(n^2 - 1)/12$.

Question 19. Is there a nice characterization of the convex polytope whose vertices correspond to the order- n Latin squares?

Two references pertaining to Latin squares arising in the context of familiar polytopes are [3, 8]. In Figure 3 we illustrate the dual of the polytope for $n = 3$ as visualized by Polymake [9] and POV-Ray [12] (the dual was chosen as we found it to be less visually confusing). Analogous questions could be asked for other combinatorial sets such as normal magic squares or Sudoku boards.

7.2. Properties of the coordinates. Given a Latin square, there are numerous transformations of it that will lead to new Latin squares. For instance, we might rotate or reflect the square around an appropriate axis or permute rows or columns. Or, if there is a 2×2 subarray of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (an *intercalate*), then we can exchange the positions of these a 's and b 's to get a new Latin square.

Or we can consider various *conjugates* of the Latin square by permuting the triples (i, j, x_{ij}) . It would be interesting to understand these operations in terms of the coordinates instead.

Alternatively, we could investigate operations on coordinates that lead to new Latin squares. For example, notice that for the order-3 Latin squares, if we consider the coordinate vectors up to sign, there are only two possibilities: $(0, \pm 1/2, \pm 1/2, 0)$ and $(\pm 1/4, \pm 1/4, \pm 1/4, \pm 3/4)$. For the 161,280 order-5 Latin squares, there are only 4,665 possibilities; each equivalence class has at least 16 elements. It is not clear how these equivalences are related, in general, to the transformations considered in the previous paragraph. Hopefully further investigation will shed light on issues such as the observation that the number of Latin squares is divisible by a surprisingly high power of 2 (see [1, 11, 10]).

Remark 20. A Sage worksheet containing code to construct the orthogonal bases described in this paper can be found at the author's web page [15].

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